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ABSTRACT

This paper discusses the development of the mathematical experiences which make up the three-term sequence of mathematics courses taken by participants in the Elementary Mathematics Project (EMP), a longitudinal study of change in preservice teachers' perceptions and beliefs about mathematics. For the mathematics courses, both content and teaching are considered. Under mathematical content, the main topics and mathematical problem solving are discussed; under doing mathematics, the topics discussed are abstraction, reasoning, unique answers, and time spent on problems. Mathematical connections are discussed with a focus on representations and applications. Instruction includes a discussion of the use of problem situations, development of community, and periodic reflections as part of the intervention. Throughout the paper, examples are given from the students' experiences in the courses. (Author/JD)

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Learning to Teach: Constructing Meaningful Understanding of Mathematical Content

Glenda Lappan and Ruhama Even

National
Center for Research
on Teacher Education

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LEARNING TO TEACH: CONSTRUCTING MEANINGFUL UNDERSTANDING OF MATHEMATICAL CONTENT

Glenda Lappan and Ruhama Even

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Abstract

This paper discusses the development of the mathematical experiences which make up the three-term sequence of mathematics courses taken by participants in the Elementary Mathematics Project (EMP). EMP is a longitudinal study of change in preservice teachers' perceptions and beliefs about mathematics. For the mathematics courses, the authors consider both content and teaching. Under mathematical content they discuss the main topics and mathematical problem solving; under doing mathematics they discuss abstraction, reasoning, unique answers, and time spent on problems. Mathematical connections are discussed with a focus on representations and applications. Instruction includes a discussion of their use of problem situations, development of community, and periodic reflections as part of the intervention. Throughout the paper, examples are given from the students' experiences in the courses.

LEARNING TO TEACH: CONSTRUCTING MEANINGFUL UNDERSTANDING OF MATHEMATICAL CONTENT

Glenda Lappan and Ruhama Even[1]

To teach the arithmetic-driven curriculum of the past, one needed little more than computational skill with the standard algorithms and a text to provide practice. That is no longer the case. To prepare a teacher dedicated to helping children think mathematically requires a very different experience with mathematics than the traditional college course for elementary teachers. With this in mind, a series of innovative mathematics courses, mathematics education courses, and field experiences has been developed for undergraduate education majors in Michigan State University's Academic Learning Program.[2] The mathematics content intervention comprises a sequence of three mathematics courses. While the mathematics content of the courses emphasizes the integration and connection of mathematics as a goal, each course highlights a different area of mathematics. These three areas were number theory, geometry, and probability and statistics.

As project members set out to design the mathematical component for the intervention, we identified the obstacles to change that we were likely to face--the beliefs and dispositions that these students bring with them as a result of 14 years of mathematics education. Our students tend to believe that (a) the elementary math curriculum is driven by computational skill as the major goal, (b) mathematical knowledge is rule-bound and not connected, (c) teaching is telling and learning is memorizing. These beliefs and dispositions are not consistent with a modern set of goals for the study of mathematics nor the needs of students. In order to challenge these beliefs and dispositions, the mathematics experience at the university has to cause the students to examine their fundamental beliefs about such questions as: What is mathematics? What does it mean to know mathematics? What mathematics do elementary school children need to study? How do we make decisions

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[1]Glenda Lappan is a professor in the Department of Mathematics at Michigan State University. She is on leave for a year at the National Science Foundation in Washington, D.C., to serve as program director for Teacher Preparation. She is a senior researcher with the National Center for Research on Teacher Education. Ruhama Even recently finished her doctorate in mathematics education at MSU. She has returned to Israel where she will join the staff of the Weizmann Institute in Rehovot. While at MSU she was a graduate assistant/curriculum developer on the Elementary Mathematics Project.

[2]The Elementary Mathematics Project is a longitudinal research project studying the change in preservice teachers perceptions and beliefs about mathematics, what it means to know mathematics, and how mathematics is learned. The project is studying students who are enrolled in the elementary part of the Academic Learning Program. Perry Lanier directs the project. Glenda Lappan is associate director of the project and principal designer and instructor for the sequence of mathematics courses. Pamela Schram and Sandra Wilcox are the project's researchers. Ruhama Even participated in the conceptualization and development of the sequence of mathematics courses and the research instruments. The Academic Learning Program at Michigan State University is an alternative teacher education program designed for highly motivated prospective teachers. The program emphasizes the development of thorough understanding of subject matters to be taught as well as knowledge of how students learn in each subject area and how to teach each subject matter effectively. Each Academic Learning student has a unique field experience which involves working with a mentor teacher and a classroom of children each term (including student teaching) over a two-year period.

about what to include in the elementary curriculum? How do children learn mathematics? What is the role of the teacher in the mathematics classroom?

In this paper we describe the sequence of the innovative mathematics courses. The paper is organized around the joint themes of mathematics and teaching. Under mathematical content we discuss the main topics and mathematical problem solving; under doing mathematics we discuss abstraction, reasoning in mathematics, unique answers, and time spent on problems. Mathematical connections is discussed with a focus on representations and applications. Instruction includes a discussion of our use of problem situations, community and periodic reflections as part of the intervention.

The Goal: Good Mathematics--Taught Well[3]

The overall goal of the three mathematics courses was "good mathematics--taught well." We believe that prospective teachers should experience the learning of good mathematics in the same way that we want them to teach their own students mathematics. Preservice teachers' own experiences provide the data they use to make sense of what mathematics is and how it should be taught. Hence, the learning environment in the three-term sequence classes had to be constructed in such a way that students experienced mathematics much as their own students might.

What do we mean by "good mathematics--taught well"? When making decisions of which mathematical ideas to pursue in the courses we asked ourselves many questions: Is this good mathematics? Is it important? What does knowing this idea enable a student to do? To what is it connected? How does it relate to the big mathematical ideas for elementary/middle school children? How does the content selected represent mathematics to the preservice teachers? Does the content require students to engage in doing mathematics--analyzing, abstracting, generalizing, inventing, proving and applying? We were most concerned with the following three facets of mathematics: mathematical content, doing mathematics, and mathematical connections. These facets are not independent from each other but rather are interrelated. Still, each one is important enough to be highlighted separately.

But "good mathematics" is not enough. Good mathematics has to be taught well. In planning the instruction for the series of math courses we were guided by three main principles: the use of problem situation, periodic reflections, and an emphasis on the community. A detailed description of the main themes that guided us in the development of the courses, both mathematical and instructional, follows.

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[3]The phrase "Good Mathematics--taught well" was first used in the Middle Grades Mathematics Project's (MGMP) Final Report to the National Science Foundation for grant #MDR 8318218. MGMP is a broad-based research, curriculum development, and teacher enhancement project dedicated to the improvement of mathematics education at the middle school level, grades 5-8.

Good Mathematics

Mathematical Content

Part of mathematical knowledge includes understanding particular topics, procedures and concepts, and the relationships among them. This is what most people usually refer to when they talk about mathematical knowledge. Since this aspect of knowledge of mathematics is fairly familiar we describe it very briefly.

Main topics. We took an overall integrated approach to mathematics, but each term had a major emphasis that allowed us to probe ideas in depth. The three main themes represent important topics in the discipline of mathematics as well as in a desired elementary/middle school curriculum (e.g., NCTM's Curriculum and Evaluation Standards for School Mathematics, 1989). The first term centered on the structure of number and number relationships. However, these ideas were approached through varied situations. Some situations were basically numerical; others geometric; and others arose from networks or the analysis of real data. In the second term the main theme was geometry, but, as before, many experiences with numbers were embedded in geometric contexts. In the last term, the emphasis shifted to data analysis, interpretation, and decision making. In each term, connections among number, geometry, probability and statistics were made.

Mathematics and problem solving. Many "mathematics for elementary teachers" textbooks start with a chapter on problem solving. After dealing with some interesting problems, the following chapters concentrate on different topics (number sets, operations on numbers, geometry, probability and statistics, to name the most common). This approach seems to imply that problem solving and mathematics are two different things: First you do problem solving and then you do mathematics. We wanted to send the message that problem solving and mathematics are not separate issues. Therefore we started with a problem which was big enough to serve as a "problem-solving" situation, but its solution was closely related to the mathematical topic to be taught--number theory. This problem--the Locker Problem--is discussed later in the paper. Throughout the three courses, we used the strategy of presenting to our students "big problems" which were related to the mathematical topic at hand. Even though the problems were related to the main topics being studied, they were still "problems" in that there was not a direct, immediate, one way to solve them. So problem solving was integrated naturally into the courses.

Doing Mathematics

Another part of mathematical knowledge includes understanding what it means to do mathematics. Many prospective elementary teachers think that knowing mathematics means mastering a given set of facts, rules and procedures (Ball, 1988; Madsen-Nason, 1988; Stodolsky, 1987; Thompson, 1984). If one sees mathematics this way, doing mathematics means recalling the appropriate fact, rule or procedure. If the situation does not look familiar, one cannot use recall, and feels unable to solve the problem. On the other hand, if the situation looks familiar, recalling facts, without understanding, may lead to misuse of

the recalled information. Belief that mathematics is more than mastering a given set of facts, rules and procedures is not sufficient. Preservice teachers need to have ideas about how to structure classrooms so that understanding can be developed. Since experience is a powerful teacher, it makes sense that these preservice teachers need to learn by experiencing mathematical ways of thinking, reasoning, analyzing, abstracting, generalizing, proving and applying in environments that model good instruction.

Abstraction. Abstraction is a major component of doing mathematics. The objects of study in mathematics are abstract creatures: numbers, shapes, functions, structures, and so forth, as opposed to objects of study in other disciplines such as matter, plants, animals or human beings. Mathematical concepts are abstract and "coming to know" in mathematics means, in many cases, abstracting, from a variety of models and situations, the important characteristics of a concept while ignoring the irrelevant ones. This approach guided us in our work. The students were provided, in many cases, with concrete materials with which to work, and were presented with various situations in which they encountered the same concept. The following example dealing with the concept of distance illustrates this point.

> *Everybody knows what a distance between two points means. For example, given the following points on a grid (Figure 1), the distance between A and B is 4 units. The distance between A and C--$\sqrt{34}$--is a little harder to calculate; the Pythagorean Theorem is needed. But suppose the grid represents a map of city streets. You are in place A and need to get to place C. Now what's the distance between A and C? Is using the Pythagorean Theorem appropriate in this case?*

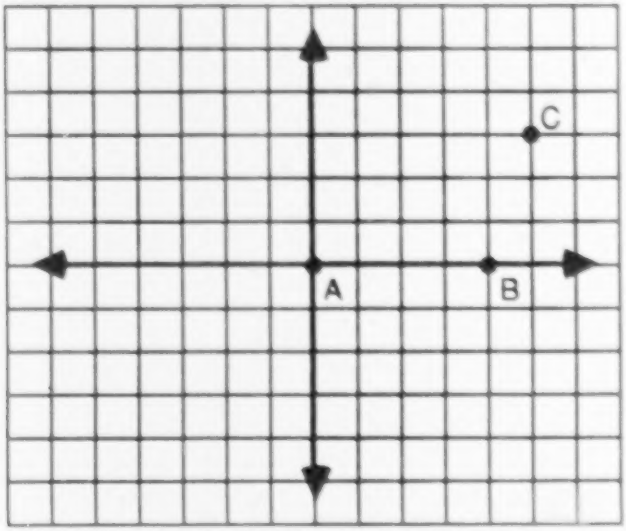

Fig. 1

Our students decided that "distance" in "taxicab geometry" should be defined to be the shortest path between two points. This definition is, of course, appropriate for distances both in Euclidian geometry as well as taxicab geometry even though the distances may differ. By having to consider the same concept in two different geometries, one of which is unfamiliar, the students needed to abstract the meaning of distance, reaching a higher level of understanding of a concept they have taken for granted.

Reasoning in mathematics. Generalization -starting from specific cases and finding a general rule--is an activity that is central to doing mathematics. One way to look at algebra, for example, is as generalized arithmetic. But generalization is not limited to algebra. Whenever we deal with relationships and look for patterns we deal with generalizations. The general rules can be described algebraically, geometrically, graphically, or verbally. Investigating a situation by checking specific cases is a very powerful strategy. Many discoveries are made by inductive reasoning. Looking at specific cases helps in understanding a situation and in seeing why a conjectured rule should hold.

Looking for patterns and describing the general rules by using inductive reasoning was an important part of all of the mathematics courses. From our experience, many prospective elementary teachers try to solve problems by searching for the "appropriate" formula. Their beliefs about what mathematics is, how one solves a math problem, and their conception of themselves in relation to mathematics shape this behavior. We wanted our students to experience inductive reasoning as a tool for solving problems in mathematics for two reasons: (a) we think that this is an important and powerful strategy in mathematics; and (b) we wanted to change their view of mathematics and what it means to do mathematics. Therefore, gathering data, checking specific examples, looking for patterns and making conjectures based on generalizations were an important part of the courses. We wanted students to see mathematics as an empirical science in order to fully appreciate mathematics as a deductive science. The following example called the Locker Problem serves as an illustration (for a thorough discussion of this problem see House, 1980).

> *In a certain high school there were 1000 students and 1000 lockers. Each year for homecoming the students lined up in alphabetical order and performed the following ritual: The first student opened every locker. The second student went to every second locker and closed it. The third student went to every third locker and changed it (i.e., if the locker was open, he closed it; if it was closed, he opened it). In a similar manner, the fourth, fifth, sixth, . . . student changed every fourth, fifth, sixth, . . . locker. After all 1000 students had passed by the lockers, which lockers were open?*

The locker problem is really a "problem." For our students there is no way to solve it by recall since the locker problem does not look like any familiar type of "story problem."

The only way to solve it is by doing mathematics. One might guess and check--popular guesses are prime number lockers, the first locker and/or the last locker. It is easy to check that the first locker remains open, but how about the last one? Prime numbers also don't seem to work (check 3 or 7, for example). It is clear that we have a *problem*. Someone in the class suggests that we see what will happen with 10 lockers. The class agrees that solving a simpler and more manageable problem might lead to some understanding of "what's going on here?" Working in small groups they "open" and "close" 10 lockers: lockers 1, 4, and 9 remained open. Then they do the same with 20 lockers--1, 4, 9, and 16 are open. Sooner or later each small group in the class has a conjecture: Either that all the open lockers are square numbers or that the differences between the open lockers are consecutive odd numbers.

Most prospective teachers are quite happy with their surprising solution and are willing to predict at this point what all the open lockers are. Since inductive reasoning is used in every day life as a mean for making predictions (e.g., Martin and Harel, 1989), most students see this stage as the final stage of the solution of the problem. But can we really be sure that the pattern continues? Why? To make sure that this is the case, deductive reasoning should be used to construct a supporting argument that is convincing.

The questions mentioned above, in addition to some others that explore the relationship between a student's number and the locker numbers visited, are assigned as homework. The next day a whole-group discussion takes place. Many students discover that the relationship between student numbers and the locker numbers visited by them can be described as the relationship between factors and multiples. Throughout the discussion it becomes clearer that open lockers are the ones that have an odd number of factors. Do all square numbers have an odd number of factors? Why? Why do nonsquare numbers have an even number of factors? Investigating these questions by exploring factor pairs for some specific numbers (e.g., factor pairs for 24 are 1 and 24, 2 and 12, 3 and 8, 4 and 6. Factor pairs for 25 are 1 and 25, 5 and 5) makes it clearer why square numbers (and only square numbers) have an odd number of factors.

How about the other conjecture? Are the differences between the open lockers consecutive odd numbers? Can we show that this is true? Proving by mathematical induction that the sum of consecutive odd numbers, starting from 1, is a square number, is not appropriate in this context. But a pictorial representation (see Figure 2) can provide a convincing argument. The number 1 is represented by one dot at the upper left corner. By adding the number 3 which is represented by three dots, we can form the number 4--a square of 2 x 2. Then, by adding the number 5 (five dots), the number nine can be formed--square of 3 x 3. One can verify that this process can continue for any given sum of consecutive odd number starting with 1. The result is always a square number.

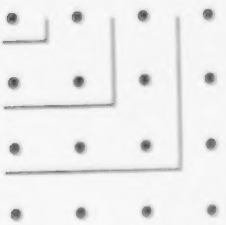

Fig. 2

From this point the class continues to explore number structure, classification of whole numbers, to discuss properties of different groups of numbers and other related ideas. The locker problem is then revisited and discussed in relation to prime numbers, least common multiple and greatest common factor.

Inductive reasoning, important as it is to mathematical activity, is not enough as an explanation for the existence of a rule nor is it a proof (unless we can check all cases--a strategy that is used more and more in modern mathematics with the power of new technology). In order to transform a conjecture to a theorem when checking all cases is not appropriate, one needs to use mathematically appropriate and acceptable ways to construct either a logical verification or a counterexample. "Deductive reasoning is the method by which the truth of a mathematical assertion is finally established" (NCTM Curriculum and Evaluation Standards for School Mathematics, 1989, p. 143). But many prospective teachers do not see the need for deductive reasoning (Even, 1989; Martin and Harel, 1989).

Providing a sound mathematical explanation was an important part of the courses. The questions "why?" and "how do you know that?" were asked often. We were not after a formal proof that uses the "appropriate" format as is often the case with high school Euclidean geometry, but rather we wanted to develop mathematical ways of thinking and reasoning at a more informal level. The observation that lockers with square numbers remain open when 30 lockers are checked does not prove that this will always be the case. Showing that square numbers have an odd number of factors and relating this to the problem does provide a convincing argument for the conjecture.

Throughout the courses we insisted that the prospective teachers reason mathematically. In contrast, as is also the case in the discipline of mathematics, some findings remained as conjectures only, without proof or refutation. This happened either because proof required tools which were too sophisticated at that stage or because we (the class--students and teacher) did not know how to go about proving it. At any rate, we

distinguished between conjectures and theorems, using mathematical reasoning but without the formalism that in so many cases hinders understanding instead of fostering it.

Unique answer. A common misconception among elementary teachers is that every math problem has one and only one answer, and there is only one way to get this answer. Not only is this a false representation of mathematics, but this way of thinking causes difficulties with the learning of mathematics. It encourages recall and memorization of the right way to solve problems instead of creativity and independent thinking. We encouraged diverse approaches and views of a problem situation throughout the courses, starting from the first meeting. The answer to the Locker Problem is an example. The solution can be described in (at least) two different ways: (a) The open lockers are all the square lockers, or (b) the difference between the nth open locker and the (n - 1)th open locker is the nth odd number or 2n - 1.

While the two answers to the locker problem are two different descriptions of the same relationship, we also presented problems which lead to completely different solutions. For example, the students were presented with the following figure (Figure 3) on the overhead projector and were asked:

> *Assume the edge of the small squares is 1 unit in length. Add squares so that the figure has a perimeter of 18. When squares are added they must meet along at least one edge of the figure exactly.*

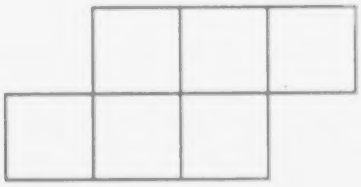

Fig. 3

After overcoming the tendency to look for a formula which will produce the perimeter of the given figure, the class counts the units around and agrees that the perimeter is 12 units. From now on each small group, using plastic unit squares to model the situation, tries to add squares to the given figure until they have a new figure with perimeter of 18. Surprisingly to many people, the fact that the perimeter is fixed does not imply that the shape of the solution figure is fixed nor that all solution figures have the same area. For example, two figures from many that work are given in Figure 4.

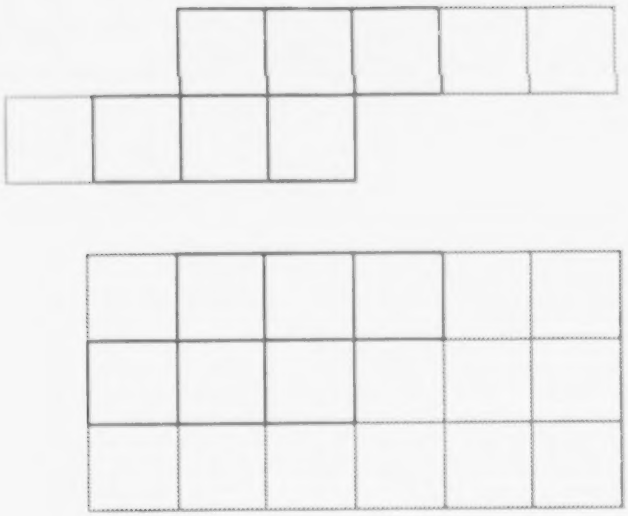

Fig. 4

Experience with problems that have more than one solution raised questions such as, "Are there any solutions that are more interesting than the others? Are there solutions that have special aspects such as largest or smallest? If so, is there a special significance to these solutions?" Since problems that arise in the real world are often ill defined or have more than one interpretation or solution, these problems show an aspect of mathematics that is very important, but rarely experienced in traditional mathematics courses.

Time spent on a problem. A common belief about solving problems in mathematics is that if one cannot solve a problem in a very short time, one will not be able to solve that problem at all (e.g., Schoenfeld, 1988). Again, this belief is shaped by the experiences one had when studying mathematics at school. If, as is usually the case with school mathematics, one is always expected to solve tens of exercises and "problems" everyday, and one is never expected to think of a problem and struggle with its solution for more than 30 minutes at most, then one is not ready to solve problems in mathematics.

In order to change this false belief about mathematics we provided many opportunities for the prospective teachers where they had to spend much more time solving one problem. We did it in two different but complementary ways. One way was to spend several class periods on the same problem. We called these problems "big problems" and used this strategy throughout the courses. But having intending teachers experience different mathematics in class is not enough. Accustomed to give up on problems very quickly, they had to be encouraged to change their behavior outside class as well.

To help our students we occasionally chose an important or interesting problem from their homework assignment and asked about its solution in class. We were not after the final answer. Rather, we wanted people to discuss their attempts, findings and difficulties in order to help them make some progress towards a solution. Still, we did not attempt to solve the problem nor to evaluate students' attempts. This process continued until the problem was solved. Spending time discussing work that had been done on the problem but without providing a solution or even an evaluation of students' attempts to solve the problem, made it clear to the students that giving up after a short trial was not "part of the game" in these courses; that they were responsible for solving the problem. The latter also implies that they can do it and therefore should try. The infinite forest problem illustrates this idea. The problem was posed as follows:

> *Suppose that you have an infinite geoboard and that on each one of the lattice points except the one at the origin there is a tree with a trunk that is only as wide as a line. You are standing on the origin. Is there a straight line path that you can take from the origin that will allow you to walk forever in the forest and not hit a tree?*

After this problem was posed it hung around and was discussed by the class for parts of several meetings before one student put forward an idea that stimulated the class to consider what it would imply if, as you walked, you did hit a tree. From this point on the solution was easy for the class. They said that if you hit a tree that implied that your path hit another lattice point. This meant that the path had a rational slope. They then constructed a length equal to the square root of 2 perpendicular to the x axis at the point (1,0). This gives a path that has an irrational slope which implies that it cannot hit another lattice point.

One of the amazing things about these long-term problems is what they often reveal to students about their own thinking. On this problem, several students put forward ideas that revealed misconceptions that were sitting there in their mathematical memory unchallenged to this point. An example was a student's notion that you could never step off the origin because the angle of any path you choose was constantly growing as you moved away from (0,0). The class probing revealed that this student and others were still confused about what it means to measure an angle! This called for a side trip into measurement to work on developing a more solid understanding of angles. These whole-class processing sessions were very instructive in helping students to see the value of big problems and of reflecting on how these problems often required us to put together ideas from several students as well as different areas to get a solution.

Mathematical Connections

Another characteristic of mathematical knowledge is rich connections (e.g., Hiebert and Lefevre, 1986). One cannot understand a mathematical concept in isolation. Connections to other concepts, procedures and pieces of information deepens and broadens one's knowledge. Two important aspects of this issue that we emphasized in our courses were the use of different representations and applications both within mathematics and between mathematics and other areas of study.

Representations. Representing ideas and problems in different ways--geometrically, verbally, numerically, algebraically or physically--allowed the students to see how different representations give different insights into problem situations (e.g., Dufour-Janvier, Bednarz, and Belanger, 1987; Lesh, Post, and Behr, 1987). Developing flexibility in representing ideas in different ways and interpreting among different representations was for us an important part of developing mathematical power. The continued work on the "perimeter 18" problem illustrates multiple representations and their power. After sharing and discussing the different solutions the class found, related questions arose:

> *What is the fewest number of squares that must be added to make the perimeter 18? What is the most number of squares that you can add and keep the perimeter 18?*

A close analysis of what happens to the perimeter when one square is added to a figure shows that if only one edge of the square touches one edge of the figure, the perimeter grows by exactly two units. For example, in the following case (Figure 5) the perimeter grew from 12 to 14.

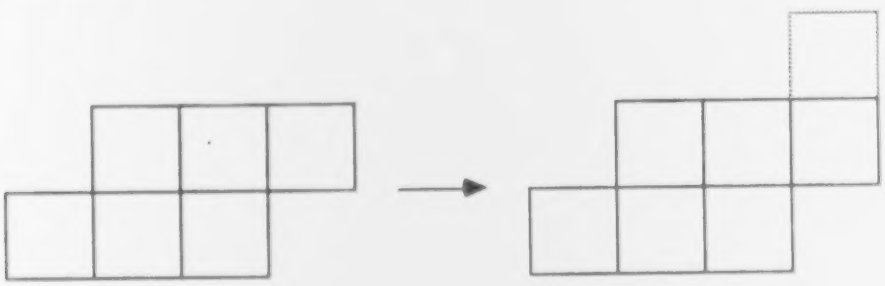

Fig. 5

If the square is added in a "corner" and two edges touch two edges of the figure, the perimeter does not change (although the area does). For example, in the following case (Figure 6) the perimeter of both figures is 14.

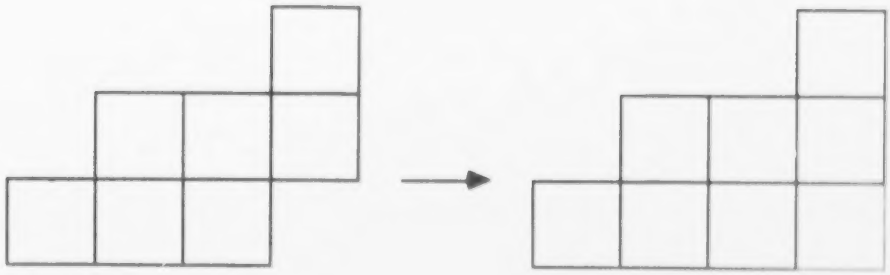

Fig. 6

Sometimes the perimeter may get smaller. It will get smaller by two units when three edges of the added square touch three edges of the figure as in the following case (Figure 7) where the perimeter went down from 18 to 16.

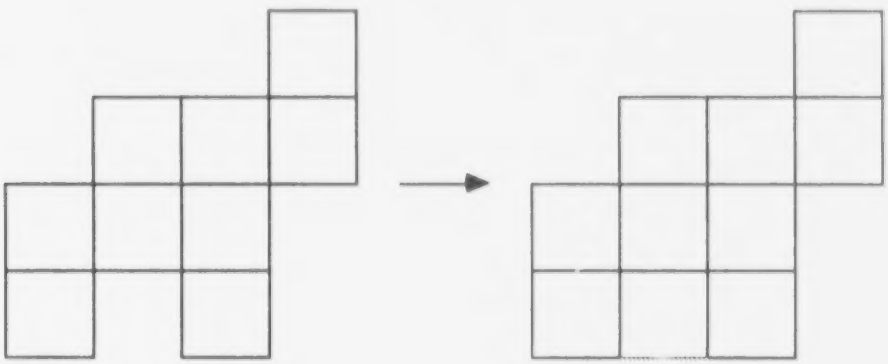

Fig. 7

Using this information it becomes clear that the shape of the resulting figure with the most squares should be a rectangle. But which one? Using tiles the students construct the following rectangles, all with perimeter of 18: 1 x 8, 2 x 7, 3 x 6, 4 x 5. They check and find out that the 4 x 5 rectangle has the most area--20. Therefore, the most squares one can add to the given figure and still get a perimeter of 18 is 14.

Is this the answer to the problem? Well, it depends on the domain in which we are working. For the given plastic tiles the 4 x 5 rectangle is the figure with most squares (area) that still has perimeter of 18. But what if we allow the dimensions of the rectangle to be

any real numbers? Further investigation of the four whole-dimension rectangles with a perimeter of 18 shows that as the bottom edge and the side edge of the rectangles become closer in length, the area grows. This leads students to conjecture that the solution figure is a square. What is the length of the square's side? Some suggestions from students were 4.5 and $\sqrt{20}$. But most students were not sure.

Graphing area vs. length of each of the rectangles (Figure 8) suggests an answer.

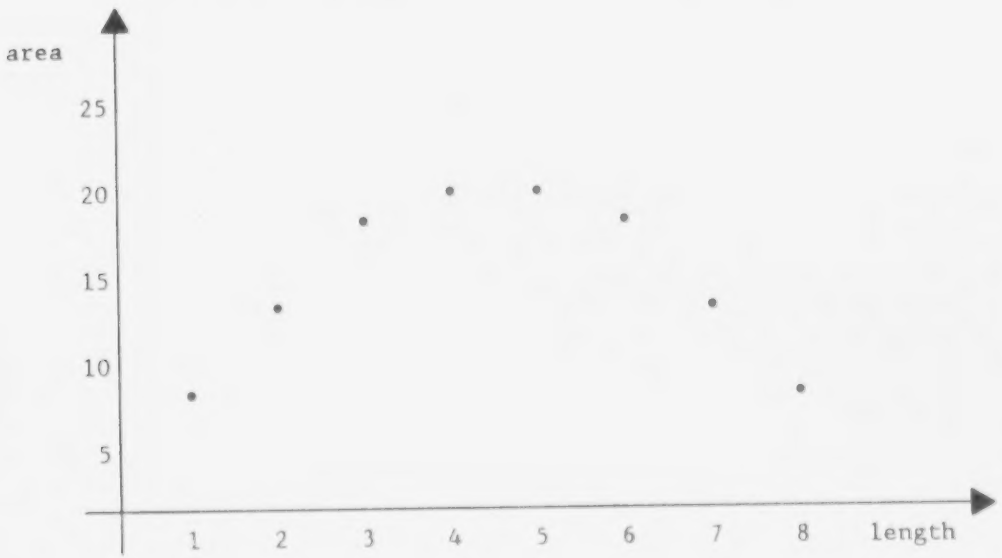

Fig. 8

The graph seems symmetric and suggests that the maximum area is midway between 4 and 5--4.5. A rectangle with perimeter 18 and length 4.5 is, of course, a square with area $4.5^2 = 20.25$. This answer seems reasonable but can we really be sure that the maximum area is obtained at 4.5? Maybe between 4 and 5 the graph goes down? Maybe it just seems to be a parabola but it is actually not?

An algebraic representation can provide a definite answer to this dilemma about where the maximum occurs, without using calculus. We did it by comparing the area of the square with perimeter 18 to the area of any rectangle with perimeter 18 (Figure 9). Let's call the width of the rectangle x, then the length is 9-x. The square has side 4.5. The square is composed of parts A and B. The rectangle of parts B and C. Since part B is common to both, we need to show that the area of part A is greater than the area of part C.

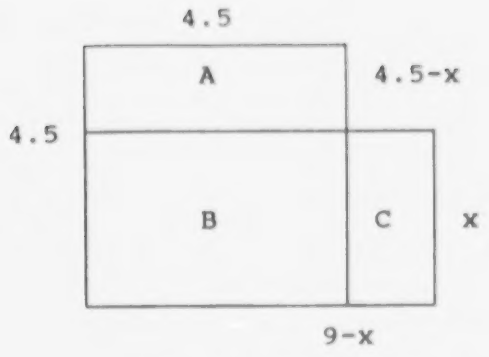

Fig. 9

Or, $4.5(4.5 - x) > x(9 - x - 4.5)$.
This can be rewritten as $4.5(4.5 - x) > x(4.5 - x)$.
Since $x<4.5$ (the width of the rectangle is shorter than the side of the square), the above inequality holds. That means that among all rectangles with perimeter 18, the square has the largest area.

The above was an example of a problem situation where moving from one representation to another contributed to a construction of richer and deeper knowledge about perimeter, area and the relationships between them; about characteristics of area of a family of rectangles--knowledge which was impossible to achieve from one representation only.

Applications. One characteristic of problem solving is application. We thought of applications as problems that require mathematical thinking in their solution and that come out of a real world situation. Such problems may call for problem solving that is as creative and as challenging as those that wear the label "problem solving." The distinction for us is the requirement of context. The situation out of which the problem arises should involve other disciplines or real world phenomena.

We paid special attention to this issue since having to apply existing knowledge in a new situation, whether inside or outside mathematics, sheds a new light on old knowledge, and creates new connections and relationships between different pieces of knowledge. We posed many problems for which students needed to integrate and apply their knowledge. For example, in the last course of the sequence the students learned new ways of looking at and interpreting data. Then, they were asked to discuss and agree upon a related set of questions that data could help answer. The class then designed a questionnaire to gather the data, planned and carried out the data gathering, analyzed the data, and organized the data for presentation of what the data said about their original problem. The class decided

that they wanted to know something about the typical MSU female and male student. They stated their questions as: Who are you Mr. MSU? and Who are you Ms. MSU?

--Taught Well

"Good mathematics" is a necessary component of a desired mathematics course. But it is not sufficient. "Good mathematics" should be taught well. Telling students that all square numbers have an odd number of factors and therefore the square number lockers are the ones that remained open at the end wouldn't have the same learning effect as the experience we described earlier. Some of the principles that guided us were implicitly described when we talked about the mathematical aspect of the sequence. Here we would like to discuss three main aspects that characterized the instruction of these courses. These were the use of problem situations, periodic reflections, and an emphasis on the community.

Problem Situations

In the traditional mathematics curriculum, mathematical facts and procedures are often studied until mastered and then applied to a specified set of problem types. The organization of texts frequently gives the learner clues that reduce problem solving to matching a pattern in a given example. The results of this kind of mathematics education are all too often students who have computational skills but have no idea when to use these skills or what the results mean in a given context. An example from the 1983 National Assessment of Educational Progress (Carpenter, Lindquist, Matthews and Silver, 1983) illustrates this problem very well:

> *An army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed?*

About 70 percent of the students correctly divided 1128 by 36 and obtained a quotient of 31 and a remainder of 12. However, less than one third of these students concluded that the number of buses needed is 32. More than one third said that the number of buses needed is "31 remainder 12."

In our courses we took as a primary goal to embed the mathematics in situations or contexts that help give the resulting concepts, rules, or procedures meaning. Research in human learning gives support to the notion that humans process information and are more likely to be able to recall and use this information if it is contextualized. Brown, Collins, and Duguid (1989), for example, argue that

> The activity in which knowledge is developed and deployed . . . is not separable from or ancillary to learning and cognition. Nor is it neutral. Rather, it is an integral part of what is learned. Situations might be said to co-produce knowledge through activity. Learning and cognition . . . are fundamentally situated. (p. 32)

The examples given in this paper illustrate this point. The locker problem, the perimeter of 18 problem--these settings became an often-used way of referring to the problem. Our students would make comments like, "This one is the Magic Johnson problem!"[4] This usually meant that the student saw a connection between the models of the Magic problem and the problem that they were trying to solve. Another result of this approach was the complete absence of comments like, "Why do we need to know this?" "What's this good for?" Students develop a very different notion of what mathematics is about if they are constantly confronted with situations from which mathematics arises rather than being given the record of other peoples rules and algorithms in an abstract form (Dewey, 1904). The NCTM Curriculum and Evaluation Standards for School Mathematics (1989) has two overall goals for students: (a) to learn to value mathematics and (b) to become confident their ability to do mathematics. Situated mathematics can contribute to each of these goals by presenting the students with interesting and meaningful mathematics in context as well as being open to various solutions and therefore enables different students to reach different levels of solutions.

Community

While communication of mathematical ideas is an important part of the experiences that all students should have in mathematics classrooms, for intending teachers this seemed to us to be critical. These students of ours needed to learn mathematics, but they also needed to become sensitive to the role of communication in clarifying one's thoughts and in expanding one's repertoire of ways of thinking. In addition, we wanted our students learn to listen to others and to try to make sense of their ideas. This led us to structure our classroom as a community of learners with considerable responsibility for judging, validating, and helping others. The teacher was not the one who gave final verification that the ideas put forward were the "correct" ones. This was the responsibility of the whole group.

The teacher's role was to pose interesting mathematical tasks for the students to consider individually, in small groups, and as a whole class. She also asked questions that helped the class to learn to value convincing arguments and to demand that mathematics make sense. At times in the whole-class processing of a problem, the instructor's role was relegated by the class to observer. The class took over and directed the discussion themselves. One particularity vivid example of the class assuming responsibility for processing the mathematics was the day the lights went out. The room in which the class met was an interior room with no windows. The class was involved in an intense discussion of the following probability problem (Figure 10) when there was instant darkness.

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[4]The Magic Johnson problem was a long-term comparison of Magic's salary with a rookie who agrees to be paid $1 the first year, $2 the second year, $4 the third, and so on with the rookie's salary doubling each year. In this problem students saw how quickly exponential growth can overcome other types of growth.

Three students are spinning to get purple (red and blue) on the given spinners. Mary chooses to spin twice on Spinner A; John chooses to spin twice on Spinner B; and Susan chooses to spin first on Spinner A and then on Spinner B. Who has the best chance of getting a red and a blue?

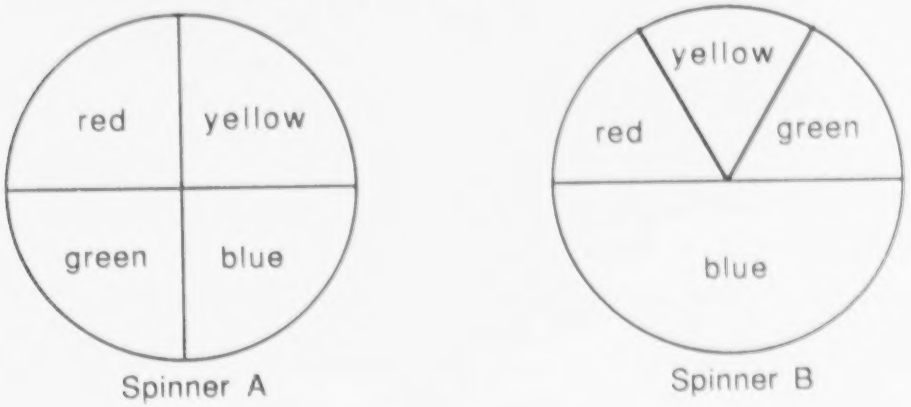

Fig. 10

The student in front of the class immediately said. "Let's practice our visualization. Be very descriptive in telling us about how you solved the problem." The discussion went on for several minutes with one student after another talking about their solution. When one very interesting suggestion was made, the class decided that they needed to see what the student had in mind. They moved the blackboard out into the foyer and gathered around to examine their classmate's idea. The student suggested that each of Mary's, John's, and Susan's spins be analyzed with a grid. She suggested that the analysis for each spin be drawn on a sheet of transparency paper (Figure 11) and then superimposed to show the results of the two spins. Figure 12 shows the suggested analysis for Susan.

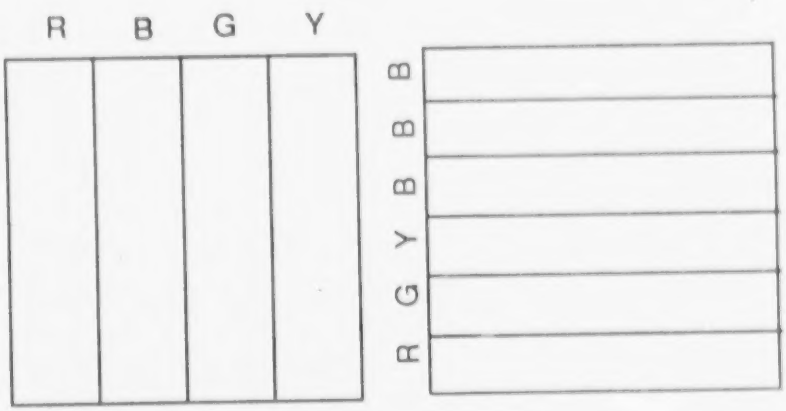

Fig. 11

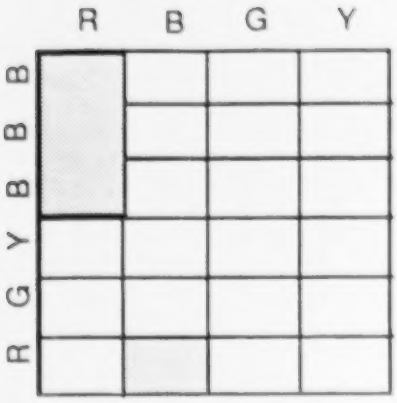

Fig. 12

The student argued that this shows the independence of the spins and the product of the probabilities in a very concrete fashion. At this stage the class broke out into applause. For a full 20 minutes the instructor had said not a word, and yet the class was fully engaged, on task and in full control of verifying the mathematics put foreword.

Creating a classroom that had this aspect of community was not a simple task. Our students' history dictated to them what math class should look like. Moving them to a new, desired, active role was very difficult. It was the long-term involvement with these students that allowed us to move them from frustration at wanting "the answer" to not only an acceptance that this was different and the demands were different, but to being unwilling to "take the teacher's word". The students have changed during the time from the first term to the third one. They took responsibility for solving problems and did not consider the teacher as the authority of right answers. Asking the teacher to verify their answers or provide the right answers, which was very common in the first term, became rare during the last term. These students insisted on understanding--on making sense of the mathematics. They had learned to value small-group work, individual effort and the power of the community of the class as a whole in resolving what to accept as valid in our growing repertoire of mathematical knowledge.

Periodic Reflections

Even if one does not think of mathematics as an arbitrary collection of bits and pieces of facts and procedures, but rather takes an overall integrated approach to mathematics, one cannot integrate everything at once. One may think of mathematics as a large picture (this does not imply that we think of mathematics as static!). In order to study the picture one needs to look at the picture very closely to see all the rich detail. However, as one gets closer, one sees more details but no longer sees the whole picture.

Therefore, after a close examination of a specific and meaningful part of the picture, the time comes to step back and look at how all the details are connected to create that specific piece of the picture. Then one should step back more and look at the whole picture from a distance, examining the relationships between the studied part and other parts of the picture as well as the whole picture.

In the mathematics courses this integrating reflection was a regular feature. While questions were asked on a daily basis that focused the students attention on connections to what had been studied and to what was coming up, we also took the time to reflect periodically as a group on the mathematics being studied. This took the form of creating concept maps of domains of knowledge, generating lists of the current working "theorems" and the conjectures that still remained to be supported or refuted, and developing different forms of representation of ideas and concepts that we had studied along with considering what each representation did to help us understand or explain a problem situation. These reflection times often included looking at the kinds of situations that the ways of thinking we had developed would likely help us solve. This took us into the realm of the real world and problem situations that are an important part of our society. To summarize, these reflection periods were to support the development of well integrated, connected knowledge of mathematics. They were of a very generative nature. The goal was not to produce a list of facts or procedures studied; it was to find new ways to organize and conceptualize the mathematical experiences that the class as a group had shared.

Conclusion

The overall purpose of school includes preparing young people for full participation in the society and culture of the modern world. Examining the role of mathematical thinking in our society forces us to reconsider the existing goals of the mathematics curriculum K-12. In particular, the goals of the elementary mathematics curriculum that center on developing computational proficiency with paper and pencil algorithms must change. Mathematics is a dynamic cultural invention that grows and changes as the needs and interests of society evolve. In the modern world this evolution of mathematical knowledge and society's dependence on mathematical ideas has become a revolution. Spurred by the invention of computing devices that make approaches to mathematics possible that were unthought of in the past, there has been a veritable explosion of mathematical thought and invention. This change in mathematics has mirrored a change in our society and culture that makes the mathematical currency of the modern world the skill and disposition to see the world mathematically--to create mathematical models of problem situations, to manipulate these models (often with the aide of a computer), and to interpret the results as they relate to the original problem.

The NCTM's new publication *Curriculum and Evaluation Standards for School Mathematics* (1989) describes a K-8 mathematics curriculum that is conceptually oriented, actively involves children in doing mathematics, and emphasizes the development of children's mathematical thinking and reasoning abilities. According to the *Standards*, a desired elementary/middle school curriculum should also emphasize the application of mathematics and include a broad range of content. How can we achieve this change in school mathematics? There is not, of course, a simple answer to this question, nor is it a new issue. The following excerpt, taken from Dienes's (1960) book *Building up Mathematics*, is as relevant now as it was 30 years ago:

> Let us face it: the majority of children never succeed in understanding the real meanings of mathematical concepts. At best they become deft technicians in the art of manipulating complicated sets of symbols, at worst they are baffled by the impossible situations into which the present mathematical requirements in schools tend to place them. (p. 13)

From the beginning of this century, Moore (1903/1926), Brownell (1935, 1947), Van-Engen (1953), Bruner (1960, 1961), Dienes (1960), Biggs and McLean, (1969), Skemp (1978) and others called for change in school mathematics. Still, the way mathematics is taught at school has changed very little.

The difficulty in making such fundamental change has been documented by many modern researchers (Joyce and Showers, 1981; McLaughin and Marsh, 1978). While a desired elementary mathematics curriculum should include new content as well as refocusing old content, changing the emphasis of the mathematical content is not enough. Good curriculum materials are necessary. But curriculum materials are not sufficient, as we can learn from the unsuccessful 1960s "new math" and 1970s individualized instruction reforms (e.g., Erlwanger, 1973; Fey, 1978). The teacher has a key role in setting mathematical goals and creating a classroom environment in which these goals are pursued (Romberg, 1988; Shulman, 1986). The routine of math class instruction as described, for example, by Welch (1978) needs to change. Going over the previous day's homework, giving a brief explanation of new material and moving around the room answering questions as students work individually on the homework, will not make the desired change in school mathematics even when done with new content. As emphasized in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), teachers "need to create an environment that encourages children to explore, develop, test, discuss, and apply ideas. They need to listen carefully to children and to guide the development of their ideas" (p. 17).

Change in teaching and learning depends heavily on the teacher. On the other hand, teaching, by its very nature, includes fundamental barriers to change (Cohen, 1988). Teaching, Cohen says, is a practice of human improvement where one human being tries

to improve the ideas, capacities, emotional states, or organization of others. Practices of human improvement are hard to manage. Therefore, most practitioners and clients tend toward conservative strategies. Most practitioners of human improvement, such as therapists and organization consultants, have some protection--they can choose their clients and are not expected to succeed without the cooperation of their clients. But teachers face the internal problems of teaching with little or no such protection. Therefore, the tendency toward conservative approaches to practice is even stronger. Cohen concludes that the internal problems of teaching that are the result of teaching being a practice of human improvement, compounded by various external conditions such as finance and organizations, cause great difficulties for teachers attempting to implement instructional reforms.

While substantial change in teaching is difficult to achieve, there are some changes that lend themselves to implementation, such as, "effective teaching" strategies. These strategies are aimed at "methodological refinement" (Aronowitz and Giroux, 1985): They provide an algorithmic approach to improving instruction--therefore are easy to adopt; they do not challenge existing authority structures in classrooms and schools. Still, using "effective teaching" strategies may change teacher behavior in some way, but by themselves do not cause teachers to teach for understanding nor do they make student learning meaningful. Past unsuccessful reforms and the internal and external barriers to change in mathematics curriculum and teaching show us that substantial change in school mathematics cannot occur easily. A good preservice education for teachers is a necessary (although not sufficient) aspect of learning to teach in ways that will enable teachers to create new desirable learning environments for students. To be effective the mathematical experiences must cause preservice teachers to build powerful mathematical schemas and to examine their deeply held beliefs about mathematics as a discipline, how it is learned and what the role of the teacher is.

Recognizing that change in beliefs and practices is very difficult to effect, we have worked from the premise that teachers need what we want for students. If students in elementary/middle school are to learn in environments that support the development of mathematical power as described in the *Curriculum and Evaluation Standards*, teachers themselves need to know mathematics and experience learning in ways that build a deep and flexible understanding of what mathematics is and what it means to do mathematics. As an intervention, we developed an entire coordinated program for the Academic Learning students which centered around the three mathematics courses described. Some of the results of the overall study are reported in Schram, Wilcox, Lanier, and Lappan, 1988, 1989). This paper is intended to give a picture of the development process and of the kind of experiences that the courses provided for our students. We hope that it will be of benefit to others that engage in research and development of preservice teachers of mathematics.

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